B.sc(mathH)part3 paper6

Topic:centre of a group

Definition : *The set Z of all self conjugate elements of a group is called the centre of G.*

i.e., $Z = \{z \in G : zx = xz \ \forall \ x \in G\}$.

Theorem *The centre Z of a group G is a normal sub group of G.*

Proof : Centre Z of G is a sub group.

Let $z_1, z_2 \in Z$, then Z will be a sub group if $z_1 z_2^{-1} \in Z$.

If $\quad z_1 \in Z \Rightarrow z_1 x = x z_1 \ \forall \ x \in G$

and $\quad z_2 \in Z \Rightarrow z_2 x = x z_2 \ \forall \ x \in G$.

Now $\quad z_2 x = x z_2 \Rightarrow z_2^{-1} (z_2 x) z_2^{-1} = z_2^{-1} (x z_2) z_2^{-1}$

$\Rightarrow x z_2^{-1} = z_2^{-1} x \ \forall \ x \in G \Rightarrow z_2^{-1} \in Z$.

We shall now show that $z_1 z_2^{-1} \in Z$.

$x\ (z_1 z_2^{-1}) = (x z_1) z_2^{-1} = (z_1 x)(z_2^{-1}), \quad \because \ z_1 \in Z$

$= z_1 (x z_2^{-1}) = z_1 (z_2^{-1} x), \quad \because \ z_2^{-1} \in Z$

$= (z_1 z_2^{-1})\ x.$

Since $x (z_1 z_2^{-1}) = (z_1 z_2^{-1}) x \ \forall \ x \in G$, it follows that

$z_1 z_2^{-1} \in Z$ whenever $z_1, z_2 \in Z$.

Hence Z is a sub group of G.

Z is a normal sub group.

Let $\quad z \in Z \Rightarrow zx = xz \ \forall \ x \in G$.

$\therefore \quad x z x^{-1} = (xz) x^{-1} = (zx) x^{-1} = z \in Z$.

$\therefore \quad z \in Z, x \in G \Rightarrow x z x^{-1} \in Z$

and hence by definition Z is a normal sub group.

Theorem *o . If Z be the centre of a group G and $a \in G$, then $a \in Z$ if and only if $N(a) = G$. If G is finite then $a \in Z$ if and only if $o(N(a)) = o(G)$.*

Proof : $\quad Z = \{z \in G : xz = zx \ \forall \ x \in Z\}$

$N(a) = \{x \in G : ax = xa\}$.

If $a \in Z$ then $ax = xa \ \forall \ x \in G$.

Since for all $x \in G$ we have $ax = xa$. Therefore $N(a) = G$. On the other hand

$N(a) = G \Rightarrow ax = xa \ \forall \ x \in G$

$\Rightarrow a \in Z$.

Again if G be finite then,

$N(a) = G \Leftrightarrow o(N(a)) = o(G)$.

Hence $a \in Z \Leftrightarrow N(a) = G \Leftrightarrow o(N(a)) = o(G)$.

Theorem *Let Z be the centre of a group G. If $a \in Z$ then prove that the cyclic sub group {a} of G generated by a is a normal sub group of G.*

Proof : $Z = \{z \in G : zx = xz \ \forall \ x \in G\}$ and $a \in Z$ so that

$ax = xa \ \forall \ x \in G$.

Again $H = \{a\}$, therefore any element h of H is of the form a^n.

***H* is a normal sub group :** Let x be any element of G, then $x h x^{-1} x = a^n x^{-1} = (x a x^{-1})(x a x^{-1}) \ldots\ldots (x a x^{-1})$

$= (x a x^{-1})^n = (a x x^{-1})^n$, by I

$= (a e)^n = a^n$ and hence it belongs to $H = \{a\}$.

Therefore $x h x^{-1} \in H \ \forall \ h \in H$ and $\forall \ x \in G$ and hence H is a normal sub group of G.

Theorem *If o (G) = p^n where p^n is a prime number then the centre $z \neq \{e\}$.*

Proof : We know $N(a)$, $a \in G$ is a sub group of G and by Lagrange's Theorem $o\,N(a)$ is divisors of the order of group G. Hence $o(N(a))$ is a divisor of p^n where p is prime and as such $o(N(a))$ is of the form p^{n_a} where n_a is an integer of the form $0 \leq n_a \leq n$.

Again know that $o(G) = p^n = \Sigma \dfrac{o(G)}{o(N(a))} = \Sigma \dfrac{p^n}{p^{n_a}}$ where $n_a \leq n$......(1)

The summation is extended over one element a in each conjugate class.

Suppose there are exactly z elements in Z i.e. $o(Z) = z$.

Now $a \in Z \Leftrightarrow N(a) = G \Leftrightarrow o(N(a)) = o(G) \Leftrightarrow p^{n_a} = p^n \Leftrightarrow n_a = n$.

Hence there are exactly z elements in G such that

$$n_a = n \text{ i.e. } \frac{p^n}{p^{n_a}} = \frac{o(G)}{o(N)(a)} = 1.$$

Therefore the class equation (1) can be written as

$$p^n = z + \Sigma \frac{p^n}{p^{n_a}} = \text{where } n_a < n.$$

or,
$$z = p^n - \Sigma \frac{p^n}{p^{n_a}}. \qquad \ldots\ldots(2)$$

From above we observe that p is divisor of R.H.S. of (2) and hence p is a divisor of Z. Again as $e \in Z$ therefore $o(Z) = z \neq 0$ and as such z is a positive integer divisible by the prime p which implies that $z > 1$. Hence z must contain an element in addition to e i.e., $z \neq \{e\}$.